

Friday, August 29, Lec 2: Series II

Last time...

• Sequences a_k and their convergence $\lim_{k \rightarrow \infty} a_k \rightarrow a$

• Series = sequence of partial sums $s_N = \sum_{k=1}^N a_k$

• Geometric series converges absolutely when $|r| < 1$:

$$\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$$

• The series $\sum_{k=1}^{\infty} k^p$ diverges when $p \geq 0$ ← HW: $p < 0$.

Today: Testing for convergence.

• Very useful technique (from last time):

① Comparison tests

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be series of nonnegative terms with

$$a_n \leq b_n \quad \text{for all } n=1, 2, \dots$$

Then we have

$$\sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} b_n$$

• If $\sum a_n \rightarrow \infty$, then $\sum b_n \rightarrow \infty$.

• If $\sum_{n=1}^{\infty} b_n = B$, then $\sum_{n=1}^{\infty} a_n = A \leq B$
↑
finite

Idea:

$$\sum a_n \leq \sum b_n$$

- If $\sum a_n$ is big, forces $\sum b_n$ to be big.
- If $\sum b_n$ is small, forces $\sum a_n$ to be small.
- Can't say anything if $\sum a_n$ is small, or if $\sum b_n$ is big.

• This is really a theorem about absolute convergence;
given any series $\sum a_n$, the series $\sum |a_n|$ has
all nonnegative terms.

Ex:

Does $\sum_{n=1}^{\infty} \frac{1}{n!}$ converge?

• Compare: We know the geometric series $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges.

Observe:

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{n!} &= \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots = \frac{1}{1} + \frac{1}{2} + \frac{1}{3 \cdot 2 \cdot 1} + \frac{1}{4 \cdot 3 \cdot 2 \cdot 1} + \dots + \\ &\leq 1 + \frac{1}{2} + \frac{1}{2 \cdot 2} + \frac{1}{2^3} + \dots \\ &\quad \uparrow \\ &\quad 2 \leq 3 \Rightarrow \frac{1}{3} \leq \frac{1}{2} \end{aligned}$$

$$= 1 + \sum_{n=1}^{\infty} \frac{1}{2^n}$$

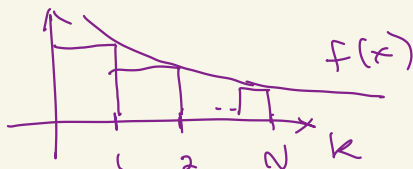
• Remember, we can always adjust finitely many
terms without affecting convergence.

What about $\sum_{k=1}^{\infty} \frac{1}{k^p}$? Well, let's think harder about comparison.

② Integral Test.

Let $f: [0, \infty) \rightarrow \mathbb{R}$ be positive and monotone decreasing.

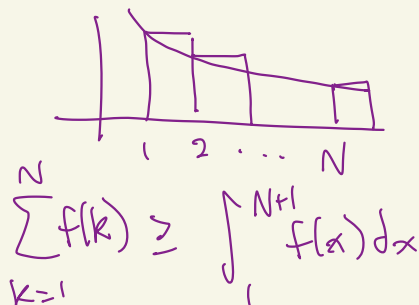
Notice:



$$\Rightarrow \sum_{k=1}^N f(k) = f(1) + f(2) + \dots + f(N) \leq \int_0^N f(x) dx$$

So if $\int_0^{\infty} f(x) dx$ converges, so does $\sum_{k=1}^{\infty} f(k)$.

Likewise,



$$\sum_{k=1}^N f(k) \geq \int_1^{N+1} f(x) dx$$

So if $\int_1^{\infty} f(x) dx$ diverges, so does $\sum_{k=1}^{\infty} f(k)$.

Thm: (Integral Test)

Let $f: [0, \infty) \rightarrow \mathbb{R}$ be positive, monotone decreasing.

$$\int_0^{\infty} f(x) dx \text{ converges} \Leftrightarrow \sum_{k=1}^{\infty} f(k) \text{ converges}$$

$$" \text{ diverges} \Leftrightarrow " \text{ diverges}$$

Remember:
you can change
finitely many
terms without
affecting
convergence!

Thm (p-series)

The series $\sum_{n=1}^{\infty} \frac{1}{n^p} \dots$

- $(p > 1)$ converges absolutely
- $(p \leq 1)$ diverges

• $p=1$ is special.
called the "harmonic series".

pf:

Homework. 

(3) Ratio Test

Thm

Let $\sum_{n=1}^{\infty} a_n$ be a series, and consider

$$r = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|.$$

- If $r < 1$, the series converges.
- If $r = 1$, inconclusive.
- If $r > 1$, the series diverges.

Ex:

Let x be a fixed real number. Show the exponential series converges:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{n!}{x^n} \cdot \frac{x^{n+1}}{(n+1)!} \right|$$

$$= \left| \frac{1}{n+1} \right| |x|$$

$$\text{So } \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{|x|}{n+1} = 0.$$

$\Rightarrow$ converges by ratio test.

So far, every test has been about absolute convergence.
(which is much more important).

There are series which converge conditionally, meaning $\sum a_n$ converges, but $\sum |a_n|$ does not (no absolute convergence).

④ Alternating Series test.

Thm:

Let $\sum a_n$ be a series such that:

i) a_n is alternating, i.e. $a_n = (-1)^n b_n$ or $a_n = (-1)^{n+1} b_n$
with $b_n \geq 0$.

ii) $\lim_{n \rightarrow \infty} a_n = 0$

iii) b_n is a decreasing sequence

Then $\sum a_n$ converges.

Note: it could be that $\sum a_n$ converges absolutely ... or not.

absolute convergence $\Rightarrow$ conditional convergence
~~*~~

Ex: Alternating Harmonic Series.

Consider $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \dots$

Then $|\frac{(-1)^n}{n}| = \frac{1}{n}$, so by p-series, it is not absolutely convergent. But

i) $\frac{(-1)^n}{n} = (-1)^n \frac{1}{n}$, with $\frac{1}{n} \geq 0$

ii) $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$

iii) since $n \leq n+1$, $\frac{1}{n} \geq \frac{1}{n+1}$, so it is decreasing.

$\Rightarrow$ By AST, converges (conditionally).

Ex:

Does $\sum_{n=1}^{\infty} (-1)^n (1 - e^{-n})$ converge?

Soln:

It doesn't converge absolutely, since
 $\sum_{n=3}^{\infty} |(-1)^n (1 - e^{-n})| \geq \sum_{n=3}^{\infty} 1 - e^{-3} = \infty$.

What about conditionally?

well, notice that

$$\lim_{n \rightarrow \infty} 1 - e^{-n} = 1$$

So this sequence looks like

$$1 - (\text{small}), \quad -1 + (\text{small}),$$

Can show it does not converge.